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LETTER TO THE EDITOR

Origin magnetisation distribution of the site-diluted Ising model on a rooted Cayley tree

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Abstract. A stochastic iterative technique is used to obtain the origin magnetisation distribution of the site-diluted Ising ferromagnet on a rooted Cayley tree with coordination number $z = 3$. This distribution shows marked morphological changes with temperature, crossing over from a broad distribution near the ferromagnetic phase boundary to a highly structured shape at lower temperatures. The large-scale structure of the intermediate- and low-temperature distributions—a superposition of similarly shaped but scaled peaks—can be understood on the basis of the contribitions of clusters distinguished by their structure close to the origin.

The site-diluted Ising model as a paradigm for diluted magnets in general has already been studied extensively (for a review see Stinchcombe 1981). On the Bethe lattice (the infinite isotropic Cayley tree) the critical behaviour of the model can be treated exactly (Young 1976). Up to now, however, little attention has been given to the properties of the local magnetisation distribution of the model. This is in contrast to the recent surge of interest in these distributions in the context of spin-glass models on the Bethe lattice (Morita 1984, Chayes *et al* 1986, Katsubara 1987, Carlson *et al* 1988, de Oliviera 1988). In this letter we present some first results on the magnetisation distribution for the origin site of a rooted Cayley tree. Due to the statistical independence of the branches of a Bethe lattice with nearest-neighbour interactions when the state of the origin spin is fixed, our results can be easily integrated to obtain the behaviour of the fully isotropic lattice. The L-level rooted tree is recursively generated by connecting σ (connectivity $\equiv z-1$) $(L-1)$-level trees at a vertex and appending a bond (see figure 1) starting from the one-level tree taken to be a single bond. In the

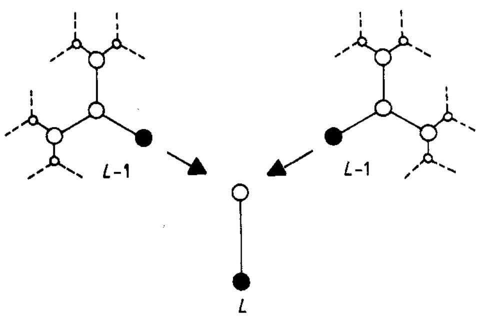

Figure 1. Recursive construction of the rooted Cayley lattice.

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case discussed here ($\sigma = 2$) one derives the following recursion equation for the random origin magnetisations m_L:

$$m_L = \varepsilon_L \varepsilon_{L-1} t \frac{m_{L-1}^{(1)} + m_{L-1}^{(2)}}{1 + m_{L-1}^{(1)} m_{L-1}^{(2)}} \tag{1}$$

where $t = \tanh(\beta J)$ is the usual Ising model thermal parameter and the ε_L are identically and independently distributed site occupation variables taking on the value $\varepsilon = 1$ with probability p and $\varepsilon = 0$ with probability $1 - p$. From here on we will assume boundary conditions chosen such that m_1 has a distribution supported on the interval $[0, 1]$ of positive magnetisations. Since the magnetisation of an unoccupied site is identically zero it suffices to consider the conditional distribution

$$\psi_L(m)\, \mathrm{d}m = \mathrm{Prob}(m_L \in [m, m + \mathrm{d}m] \,|\, \varepsilon_L = 1). \tag{2}$$

The limiting distribution ψ obtained as $L \to \infty$ will satisfy the integral equation

$$\psi(m) = p \int_0^1 \mathrm{d}m^{(1)} \int_0^1 \mathrm{d}m^{(2)}\, \psi(m^{(1)}) \psi(m^{(2)}) \delta\left(m - t \frac{m^{(1)} + m^{(2)}}{1 + m^{(1)} m^{(2)}}\right) + (1 - p)\delta(m). \tag{3}$$

From the δ constraint in the first term of (3) it is clear that m is bounded above by the largest positive root of

$$m_* = t \frac{2m_*}{1 + m_*^2} \rightarrow m_* = \begin{cases} 0 & t \in [0, \frac{1}{2}] \\ \{2t - 1\}^{1/2} & t \in [\frac{1}{2}, 1] \end{cases} \tag{4}$$

which is just the magnetisation of the undiluted case, the critical coupling $t_c(1) = \frac{1}{2}$, providing a bound on the transition point $t_c(p)$ for the diluted case. In the following we will restrict ourselves to the regime $t \geqslant t_c(1)$ and introduce the scaled magnetisations $\mu = m/m_*$. Furthermore, anticipating the trivial contribution of the finite clusters, we will look for solutions of the form $\psi(\mu) = (1 - Q)\phi(\mu) + Q\delta(\mu)$. Insertion in (3) yields for Q:

$$Q = pQ^2 + (1 - p) \rightarrow Q = \begin{cases} 1 & p \in [0, \frac{1}{2}] \\ (1 - p)/p & p \in [\frac{1}{2}, 1] \end{cases} \tag{5}$$

which indeed is the probability that the origin belongs to a finite cluster (see, e.g., Essam 1980). In view of (5) we expect non-trivial distributions only for p larger than the percolation threshold $p_c = \frac{1}{2}$ and we will leave this condition understood in the following. We are left with the equation for ϕ:

$$\begin{aligned} \phi(\mu) = u \int \mathrm{d}\mu^{(1)}\, \phi(\mu^{(1)}) \delta(\mu - t\mu^{(1)}) \\ + (1 - u) \int \mathrm{d}\mu^{(1)} \int \mathrm{d}\mu^{(2)}\, \phi(\mu^{(1)}) \phi(\mu^{(2)}) \delta\left(\mu - t \frac{\mu^{(1)} + \mu^{(2)}}{1 + (2t - 1)\mu^{(1)}\mu^{(2)}}\right) \end{aligned} \tag{6}$$

where $u = 2(1 - p)$ is the probability that the infinite cluster to which the origin belongs does *not* develop a side branch at the first site connected to it. One checks that the paramagnetic solution $\phi_0(\mu) = \delta(\mu)$ solves (6) for any t. Inserting a distribution into the right-hand side which is non-zero only in a small positive neighbourhood of zero, one finds that a solution with non-zero first moment will become stable for $t > t_c(p) = 1/2p$, which identifies the well known phase boundary of the model. The ground-state solution at $t = 1$ is also readily obtained: $\phi_1(\mu) = \delta(\mu - 1)$ implying an average magnetisation $\bar{m} = 1 - Q$ which is, as expected, equal to the fraction of sites in the infinite cluster connected to the origin. In order to solve (6) for arbitrary values of t in the ordered phase we implement the following iterative scheme. In the ith generation we have $\mathcal{N}$ magnetisations $\{\mu_{(i)}[n] \mid n=1,2,\ldots,\mathcal{N}\}$. The $i+1$th generation is created by the following process:

$$\mu_{(i+1)}[n] = \begin{cases} \mu_{(i)}[n'] & \text{with probability } u \\ t\dfrac{\mu_{(i)}[n'] + \mu_{(i)}[n'']}{1+(2t-1)\mu_{(i)}[n']\mu_{(i)}[n'']} & \text{with probability } 1-u \end{cases} \tag{7}$$

$n = 1, 2, \ldots, \mathcal{N}$ $\qquad$ n', n'' randomly chosen from $\{1, 2, \ldots, \mathcal{N}\}$.

Physically speaking, this process can be interpreted as the recursive growth of $\mathcal{N}$ realisations of systems belonging to the quenched ensemble of configurations that have a cluster connecting the origin with the surface of the tree. The index i labels the level of the roots created after the ith iteration. In order to obtain the distribution of magnetisations, the interval $[0, 1]$ is divided into M subintervals and the number of magnetisations falling into every subinterval is counted. In practice the starting distribution was chosen uniform on $[0, 1]$. The system was then allowed to equilibrate for a number N_{eq} of generations; the value of the average magnetisation $\bar{m}$ was used to monitor the equilibration. Subsequently a measurement was performed by averaging the distribution over N_{run} generations. In the examples below we chose $\mathcal{N} = 2^{15}$, $M = 2^8$, $N_{eq} = 30$ and $N_{run} = 50$ except where noted. The system we consider has site occupation probability $p = 0.75$. Figure 2 shows the distribution close to the phase boundary ($t_c = \frac{2}{3}$) at $t = 0.675$; a broad single-peaked structure. At $t = 0.825$ (i.e. at lower temperature) we find the highly structured distribution shown in figure 3; an array of peaks that apart from amplitude and width appear to share a common shape. Figure 4 presents the part of the distribution containing the two most prominent peaks obtained

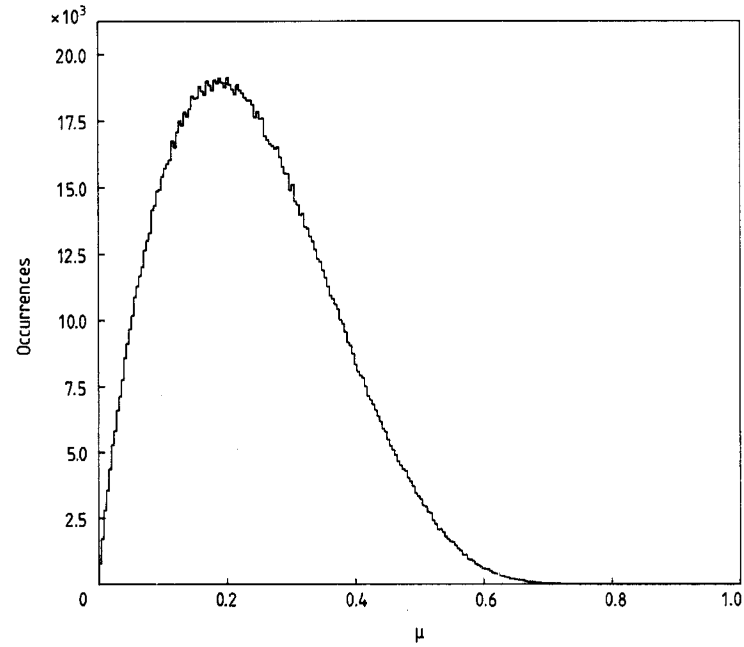

Figure 2. Distribution of the reduced magnetisation μ for $p = 0.75$ at $t = 0.675$.

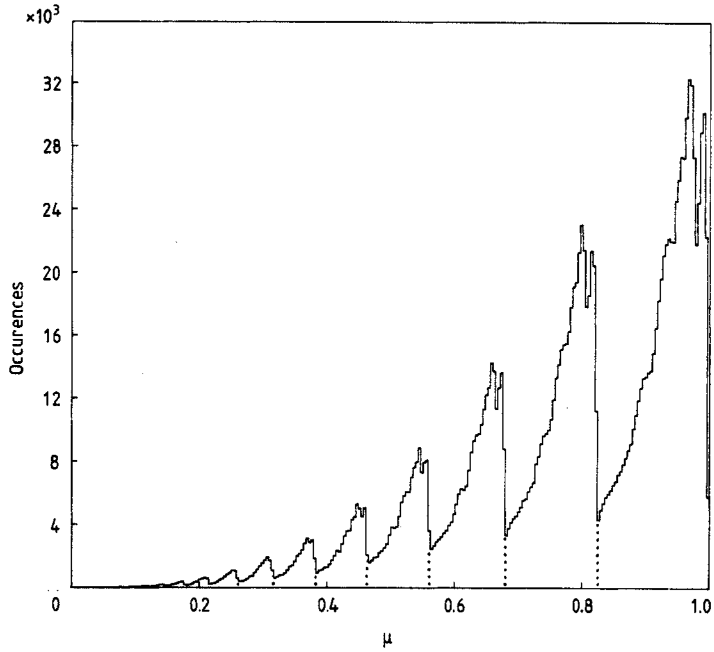

Figure 3. Distribution of the reduced magnetisation μ for $p=0.75$ at $t=0.825$. Dotted lines represent predicted locations of cutoffs.

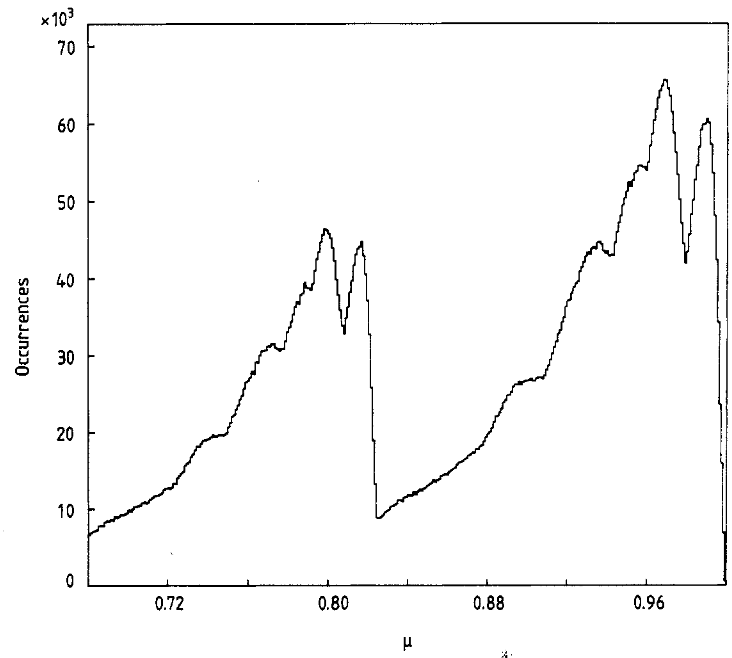

Figure 4. Detail of the distribution at $t=0.825$ obtained from a higher-resolution calculation.

from a higher-resolution calculation at the same temperature ($\mathcal{N} = 2^{17}$, $M = 2^{10}$, $N_{eq} = 50$, $N_{run} = 100$) clearly exhibiting this shape similarity. In order to explain the multiple-peak structure of the distribution, consider the contribution of the class of clusters $\mathscr{C}_n$ being all the clusters that develop their first side branch at a site n levels up from the origin. The maximum contribution to the scaled magnetisation from this class—$\mu_{max}^{(n)}$—comes from the zero-probability cluster that is fully occupied from the branching site upwards. From the recursive process (7) one easily deduces that $\mu_{max}^{(n)} = t^{n-1}$. The arrows in figure 3 show the location of these predicted cutoffs which indeed coincide with the sharp fall-off of the observed peaks. Since the structures of the clusters beyond the first branching point are identical for all $\mathscr{C}_n$ regardless of n, this analysis also supplies a heuristic for the observed shape similarity of the peaks. Moreover one predicts that the ratio of the weights of the contributions (i.e. the area under the separate peaks) to the distribution coming from two successive classes of clusters $\mathscr{C}_n$ and $\mathscr{C}_{n+1}$ should be equal to u—the non-branching probability. In the low-temperature regime, where the peaks are well separated, the validity of our analysis is of course readily checked (see figure 3). Closer to the phase boundary, however, the individual peaks have spread to such an extent that their superposition produces a smooth envelope as shown in figure 2, totally obscuring their separate contributions. The power of our analysis is further demonstrated when considering an intermediate-temperature distribution at $t = 0.75$ where the separate peaks are already evident but still have significant overlap so as to produce what seems at first sight to be a non-trivial background. Figure 5 shows the calculated distribution as well as a prediction obtained by first fitting a single Euler beta distribution to the first peak in the regime $\mu \geqslant t = 0.75$ where there is no mixing with other contributions, and then generating scaled copies of this distribution having cutoffs and relative weights given by our analysis and finally summing these contributions. In spite of the simple choice of basic fitting function we find highly satisfactory agreement with the observed distribution showing that our 'first-order' analysis indeed captures most of the large-scale features. The analysis can of course be extended by considering specific subclasses of the $\mathscr{C}_n$ in order to explain more details of the fine structure evident within the peaks in the low-temperature distributions. Finally in figure 6 we present the average magnetisation $\bar{m}$ as a function of the thermal parameter, as calculated from the distributions, showing a smooth monotonic behaviour. Our results indicate that the n delta function approximations

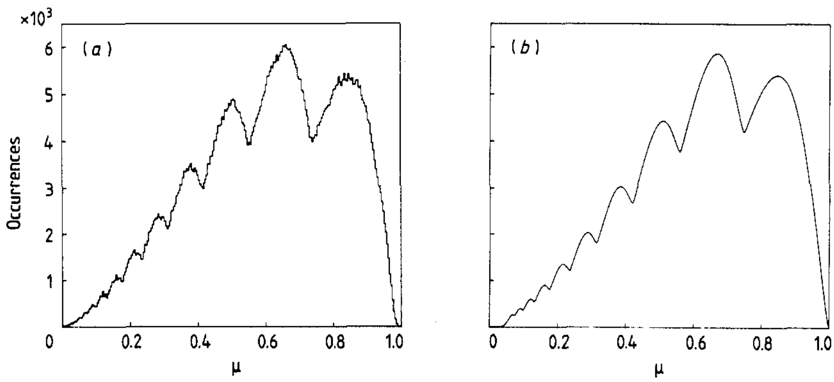

Figure 5. (a) Calculated distribution at $t = 0.75$. (b) Predicted distribution from identical but scaled contributions from the classes of clusters $\mathscr{C}_n$, $n = 1, \ldots, 10$.

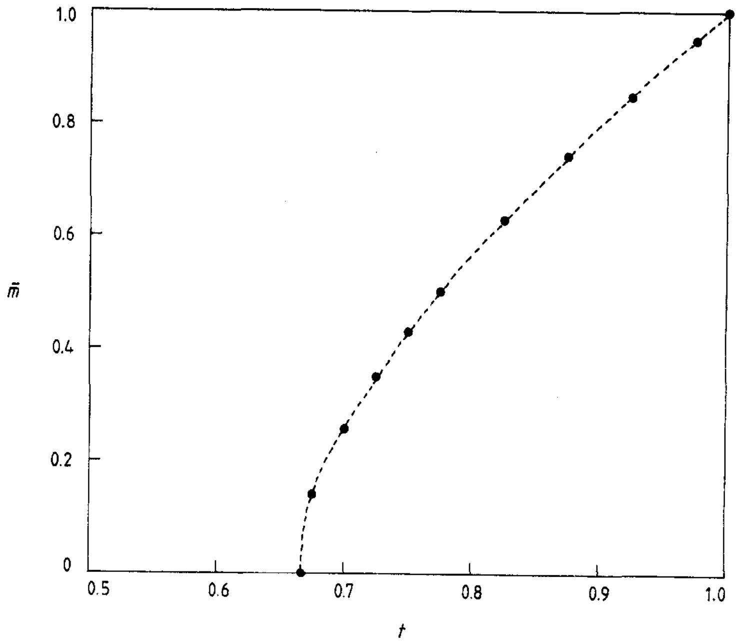

Figure 6. Average magnetisation $\bar{m}$ as a function of $t = \tanh(\beta J)$.

as discussed by Young (1976)—although probably able to reproduce the *average* magnetisation quite accurately—dramatically fail to describe the morphology of the actual distributions.

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